Roots of Unity POTD Solutions

Kenan Hasanaliyev and Raymond Feng

Week of 11/10

Monday (ARML). Find the sum of all integer values of x between 0 and 90 inclusive so that $(\cos(x^\circ) + i\sin(x^\circ))^{75}$ is a real number.

Tuesday (Moldovan TST). Let S be the set of all positive integers that have 100 digits, are divisible by 3, and only contain digits in the set $\{3, 5, 7, 9\}$. Find the remainder when the number of elements of S is divided by 29.

Wednesday (Math Prize for Girls). Compute the number of integers n between 1 and 2019 inclusive such that

$$\prod_{k=0}^{n-1}\left(\left(1+e^{\frac{2\pi ik}{n}}\right)^n + 1\right) = 0.$$

Thursday (Titu). For positive integers n, define

$$f(n) = \sum_{k=0}^{n-1}\cos^{2n}\left(\frac{k\pi}{n}\right).$$

Compute

$$\sum_{k=2}^{\infty}\frac{f(k)}{k\cdot 2^k}.$$

Friday (Titu). There exists a fraction a such that

$$\left|\frac{6z-i}{2+3iz}\right| \leq 1$$

if and only if $|z| \leq a$. Given that a can be expressed as $\frac{p}{q}$ where p and q are relatively prime positive integers, compute $p+q$.